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AREA COMPUTATION OF A POLYGON ON AN ELLIPSOID

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Abstract

A method has been developed for the area computation of a polygon on an ellipsoid. The sides of this polygon may be geodetic lines, loxodromes, great circles or a combination of these lines.

Introduction

Boundaries of exploration or exploitation concessions on the Dutch Continental Shelf are defined by parallels (loxodromes), great circles and sometimes even by arcs of a circle. In order to compute the area of such a concession an algorithm was needed that could process these types of lines. As algorithms for the geodetic line, great circle and the loxodrome had been developed already and an equal area projection is rather straightforward, it seemed a good idea to combine these elements in a step-by-step process for computing polygonal areas.

First, if the polygon is known in polar co-ordinates, the formula will be given for computation of polygonal areas in a plane.

Then the step-by-step process and the use of the mapping formulas will be explained.

Next the computation of the correction on the step-by-step process will be discussed.

And finally the mapping formulas for the Albers equal area projection will be derived. In contrast with the derivation in *Elements of map projection* [3] the formulae will be derived for computation directly from ellipsoid to cone; the step of computing the authalic (equivalent) latitude on the sphere is omitted.

Polygon Areas

If the points and interpolating points of the polygon area on the ellipsoid are equivalently projected in a plane (see Fig. 1) and if the points $P_1, P_2, \ldots, P_n$ in the plane are defined by polar co-ordinates (ρ_i, θ_i), ρ is the radius vector, θ is the vectorial angle, then the area of polygon

$$P_1, P_2, \ldots, P_n = \tfrac{1}{2}|\rho_1\rho_2 \sin(\theta_2-\theta_1) + \rho_2\rho_3 \sin(\theta_3-\theta_2) + \ldots \\ \ldots + \rho_{n-1}\rho_n \sin(\theta_n-\theta_{n-1}) + \rho_n\rho_1 \sin(\theta_1-\theta_n)| \tag{1}$$

The formula is in absolute value, so the points $P_1, P_2, \ldots, P_n$ may be entered counterclockwise or clockwise.

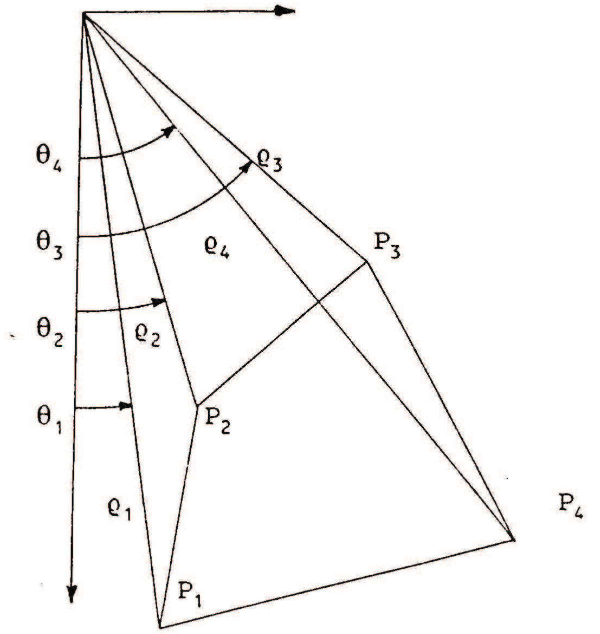

Fig. 1.

The Step-by-Step Process

Between the endpoints of the polygon interpolating points are computed along the boundaries. Then all these points are mapped in the plane with the Albers projection and the area is computed with formula (1). In general the geodetic line, the loxodrome or any other line on the ellipsoid will be curved lines after projection in a plane. To compensate for the curvature, interpolating points between the endpoints of the sides of the polygon will have to be computed. If a side of the polygon is defined by the *geodesic* between the polygon points P_i and P_{i+1} the interpolating points between P_i and P_{i+1} are computed as follows.

(1) Compute the azimuth from P_i to P_{i+1} of the geodesic, e.g. with the method of Vincenty [**4**].

(2) The distance from P_i to the interpolating point is the chosen step size.

(3) Then the coordinates of the first interpolating point are computed directly from azimuth and distance.

(4) The next interpolating point is computed in the same way from P_i with the same azimuth and twice the step size.

(5) This process is repeated until P_{i+1} is reached.

If a side of the polygon is defined by the *loxodrome* between the endpoints the same procedure is applied, except that the direct solution of the loxodrome is used for computing the interpolating points, for example with the method of Bowring [**1**]. And for *great circles* the same process can be applied. As a matter of fact the same process may be used for any type of line as long as interpolating points can be computed. Even polygons drawn on a map in any projection can be computed by using the inverse mapping formulas for transformation of the endpoints and interpolating points to the ellipsoid, and then with the Albers projection back to the plane.

Mapping Formulae

The equivalent Albers projection employs a cone intersecting the ellipsoid at two parallels known as the standard parallels. In general the projection is used to represent a certain area, and consequently the standard parallels are chosen accordingly. In this case the projection is used for computation purposes only, and no relevant difference in the area computation caused by the choice of the standard parallels was found.

The meridians are straight lines in this projection, so if the endpoints of a side are on the same meridian, no interpolating points need to be computed.

The mapping equations for a point P given in latitude and longitude to P′ in vectorial angle and radius vector or $P(\varphi, \lambda)_{\text{ellipsoid}} \to P'(\rho, \theta)_{\text{plane}}$ are:

$$\begin{aligned} \rho &= \sqrt{\left[\rho_1^2 + \frac{2a^2}{n}\,(1-e^2)(\beta_1-\beta)\right]}, \\ \theta &= n\lambda, \end{aligned} \tag{2}$$

where ρ is the radius vector, θ is the vectorial angle, n is the mapping factor for λ, λ is the geodetic longitude, φ is the geodetic latitude, φ_1 is the geodetic latitude of standard parallel 1, φ_2 is the geodetic latitude of standard parallel 2 and a is the equatorial radius of the ellipsoid.

$$\beta = \sin\varphi\left(1 + \frac{2e^2}{3}\sin^2\varphi + \frac{3e^4}{5}\sin^4\varphi + \ldots\right) \tag{3}$$

$$n = \frac{\left(\dfrac{\cos^2\varphi_1}{1-e^2\sin^2\varphi_1} - \dfrac{\cos^2\varphi_2}{1-e^2\sin^2\varphi_2}\right)}{2(1-e^2)(\beta_2-\beta_1)}. \tag{4}$$

For standard parallel 1:

$$\rho_1 = \frac{a\cos\varphi_1}{n(1-e^2\sin^2\varphi_1)^{\frac{1}{2}}}. \tag{5}$$

The complete mathematical theory will be treated later on.

Accuracy and Correction

As the sides of the polygon are approximated by interpolating points, which after projection are as it were connected by straight lines for the area computation, an error is introduced, the magnitude of which depends on the step size.

Theoretically one would expect the error to decrease quadratically with a decreasing step size. Empirically this was confirmed; if we have the step sizes s_1 and s_2 with corresponding errors e_1 and e_2 and if

$$\left.\begin{aligned} s_2 &= \frac{1}{a}s_1 \\ \text{then} \qquad e_2 &= \frac{1}{a^2}e_1. \end{aligned}\right\} \tag{6}$$

Knowing this the error can be computed and used as a correction.

Suppose the true area is A. The computed area with step size s_1 is $A_c = A+e_1$. The computed area with step size s_2 is $A_c = A+e_2$.

$$\Delta A_c = (A+e_2)-(A+e_1) = e_2-e_1. \tag{7}$$

Substituting $e_2 = 1/a^2 e_1$ yields:

$$\frac{1}{a^2}e_1 - e_1 = \Delta A_c \Rightarrow e_1 = \frac{a^2}{1-a^2}\Delta A_c. \tag{8}$$

So if the area is computed twice, the second time with for instance half the step size, the error can be computed as

$$e_1 = -\tfrac{4}{3}\Delta A_c \tag{9}$$

and can be used as correction.

Using the step-by-step process with this correction method for areas – with geodesic boundaries – of any size there was no difference with the method of Danielsen [2] on the square metre level.

The area of the complete ellipsoid (Hayford) computed with the formula for the oblate ellipsoid is:

$$\text{Area} = 2\pi a^2 + \pi\frac{b^2}{e}\ln\frac{1+e}{1-e} = 510\,100\,933.858\,376\ \text{km}^2. \tag{10}$$

This area computed with the corrected step-by-step method is: area = 510 100 933.858 384 km^2. With step sizes of 500 m and 250 m, the difference was 0.000 008 km^2, or 15.7×10^{-9} p.p.m.

Mathematical Theory of the Albers Projection [3]

If a is the equatorial radius of the ellipsoid, e the eccentricity, and φ the latitude, the radius of curvature of the meridian is given in the form

$$\rho_m = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}}, \tag{11}$$

and the radius of curvature perpendicular to the meridian is equal to

$$\rho_n = \frac{a}{(1-e^2\sin^2\varphi)^{\frac{1}{2}}}. \tag{12}$$

The differential element of length of the meridian is therefore equal to the expression

$$dm = \frac{a(1-e^2)\,d\varphi}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}}, \tag{13}$$

and that of the parallel becomes

$$dp = \frac{a\cos\varphi\,d\lambda}{(1-e^2\sin^2\varphi)^{\frac{1}{2}}}, \tag{14}$$

in which λ is the longitude in radians.

The element of area on the ellipsoid is thus expressed in the form

$$dS = dmdp = \frac{a^2(1-e^2)\cos\varphi\, d\varphi\, d\lambda}{(1-e^2\sin^2\varphi)^2}. \tag{15}$$

We now wish to determine an equal-area projection of the ellipsoid in the plane. If ρ is the radius vector in the plane, and θ is the angle which this radius vector makes with some initial line, the element of area in the plane is given by the form

$$dS' = \rho\, d\rho\, d\theta. \tag{16}$$

ρ and θ must be expressed as functions of φ and λ, and therefore

$$d\rho = \frac{\partial\rho}{\partial\varphi}d\varphi + \frac{\partial\rho}{\partial\lambda}d\lambda \tag{17}$$

and

$$d\theta = \frac{\partial\theta}{\partial\varphi}d\varphi + \frac{\partial\theta}{\partial\lambda}d\lambda. \tag{18}$$

We will now introduce the condition that the parallels shall be represented by concentric circles: ρ will therefore be a function of φ alone, or

$$d\rho = \frac{\partial\rho}{\partial\varphi}d\varphi. \tag{19}$$

As a second condition, we require that the meridians be represented by straight lines, the radii of the system of concentric circles. This requires θ to be independent on φ, or

$$d\theta = \frac{\partial\theta}{\partial\lambda}d\lambda. \tag{20}$$

Furthermore, if θ and λ are to vanish at the same time (if $\lambda = 0$ then $\theta = 0$ and vice versa), and if equal differences of longitude are to be represented at all points by equal arcs on the parallels, θ must be equal to some constant times λ, or

$$\theta = n\lambda, \tag{21}$$

in which n is the required constant. This gives us

$$d\theta = n\,d\lambda. \tag{22}$$

By substituting these values in the expression for dS', we get

$$dS' = \rho\frac{\partial\rho}{\partial\varphi}n\, d\varphi\, d\lambda. \tag{23}$$

Since the projection is to be equal-area,

$$dS' = -dS. \tag{24}$$

The minus sign is explained by the fact that ρ decreases as φ increases, or

$$\rho\frac{\partial\rho}{\partial\varphi}n\, d\varphi\, d\lambda = -\frac{a^2(1-e^2)\cos\varphi\, d\varphi\, d\lambda}{(1-e^2\sin^2\varphi)^2}. \tag{25}$$

By omitting $d\lambda$ we find that ρ is determined by the integral

$$\int_0^\varphi \rho \frac{\partial \rho}{\partial \varphi} d\varphi = -\frac{a^2(1-e^2)}{n} \int_0^\varphi \frac{\cos\varphi \, d\varphi}{(1-e^2\sin^2\varphi)^2}. \tag{26}$$

If R represents the radius ($\rho(0)$) for $\varphi = 0$, this becomes

$$\rho^2 - R^2 = -\frac{2a^2(1-e^2)}{n} \int_0^\varphi \frac{\cos\varphi \, d\varphi}{(1-e^2\sin^2\varphi)^2}. \tag{27}$$

Because

$$\frac{1}{(1-e^2\sin^2\varphi)^2} = 1 + 2e^2\sin^2\varphi + \frac{2(2+1)}{2!}(e^2\sin^2\varphi)^2 + \frac{2(2+1)(2+2)}{3!}(e^2\sin^2\varphi)^3 + \ldots \tag{28}$$

we get:

$$\rho^2 - R^2 = -\frac{2a^2(1-e^2)}{n} \int_0^\varphi \times (\cos\varphi + 2e^2\sin^2\varphi\cos\varphi + 3e^4\sin^4\varphi\cos\varphi + 4e^6\sin^6\varphi\cos\varphi)\, d\varphi \tag{29}$$

or

$$\rho^2 - R^2 = -\frac{2a^2(1-e^2)}{n}\left(\sin\varphi + \frac{2e^2}{3}\sin^3\varphi + \frac{3e^4}{5}\sin^5\varphi + \ldots\right) \tag{30}$$

or

$$\rho^2 = R^2 - \frac{2a^2}{n}(1-e^2)\beta; \quad \text{general equation for } \rho. \tag{31}$$

The two constants n and R are as yet undetermined.

Let us introduce the condition that the scale shall be exact along two given parallels. On the ellipsoid the length of the parallel for a given longitudinal difference λ ($\lambda = \lambda_2 - \lambda_1$) is equal to the expression

$$\int_{\lambda_1}^{\lambda_2} \frac{a\cos\varphi}{(1-e^2\sin^2\varphi)^{\frac{1}{2}}} d\lambda \quad \text{so} \quad \mathrm{P} = \frac{a\lambda\cos\varphi}{(1-e^2\sin^2\varphi)^{\frac{1}{2}}}. \tag{32}$$

On the map the arc P is represented by

$$\rho\theta = \rho n \lambda. \tag{33}$$

On the two parallels along which the scale is to be exact, if we denote them by subscripts, we have

$$\rho_1 n\lambda = \frac{a\lambda\cos\varphi_1}{(1-e^2\sin^2\varphi_1)^{\frac{1}{2}}} \tag{34}$$

or, on omitting λ, we have

$$\rho_1 = \frac{a\cos\varphi_1}{n(1-e^2\sin^2\varphi_1)^{\frac{1}{2}}} \tag{35}$$

and

$$\rho_2 = \frac{a\cos\varphi_2}{n(1-e^2\sin^2\varphi_2)^{\frac{1}{2}}}. \tag{36}$$

Substituting these values in turn in the general equation for ρ, we get

$$R^2 - \frac{2a^2}{n}(1-e^2)\beta_1 = \frac{a^2\cos^2\varphi_1}{n^2(1-e^2\sin^2\varphi_1)} \tag{37}$$

$$R^2 - \frac{2a^2}{n}(1-e^2)\beta_2 = \frac{a^2\cos^2\varphi_2}{n^2(1-e^2\sin^2\varphi_2)}. \tag{38}$$

By subtracting

$$2n(1-e^2)(\beta_2-\beta_1) = \frac{\cos^2\varphi_1}{1-e^2\sin^2\varphi_1} - \frac{\cos^2\varphi_2}{1-e^2\sin^2\varphi_2} \tag{39}$$

on reducing, we get

$$n = \frac{(\cos^2\varphi_1/1-e^2\sin^2\varphi_1)-(\cos^2\varphi_2/1-e^2\sin^2\varphi_2)}{2(1-e^2)(\beta_2-\beta_1)}. \tag{40}$$

By substituting the value of n in the above equations, we could determine R, but we are only interested in cancelling this quantity from the general equation (31) for ρ:

$$\rho_1^2 = R^2 - \frac{2a^2(1-e^2)}{n}\beta_1. \tag{41}$$

$$\rho^2 = R^2 - \frac{2a^2(1-e^2)}{n}\beta. \tag{42}$$

By subtracting we get

$$\rho^2 - \rho_1^2 = \frac{2a^2}{n}(1-e^2)(\beta_1-\beta) \tag{43}$$

and so $$\rho = \sqrt{\left[\rho_1^2 + \frac{2a^2(1-e^2)}{n}(\beta_1-\beta)\right]}.$$

Already we have found

$$\theta = n\lambda.$$

References

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